

Fractional Models of Cosmic Ray Acceleration in the Galaxy ¹

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Аннотация

Possible formulations of the problem of cosmic rays acceleration in the interstellar galactic medium are considered with the use of fractional differential equations. The applied technique has been physically justified. A Fermi result has been generalized to the case of the acceleration of particles in shock waves in the supernovae remnants fractally distributed in the Galaxy.

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1 Introduction

The physical meaning of the fractional differential equation

$$\left[{}_0D_t^\beta + D(-\Delta)^{\alpha/2} \right] \psi(\mathbf{x}, t) = \delta(\mathbf{x})\delta_\beta(t), \quad \delta_\beta(t) \equiv t_+^{-\beta}/\Gamma(1-\beta), \quad \alpha \in (0, 2], \quad \beta \in (0, 1], \quad (1)$$

was discussed in previous work [1] in the continuous time random walk (CTRW) model in view of the problems arising when it is applied to the description of the transport of cosmic rays in the Galaxy. In this work, the CTRW approach is considered in application to the description of the acceleration (more precisely, reacceleration) of cosmic rays. In contrast to the preceding problem, this concerns the behavior of particles in the momentum space. Successive interactions (collisions) of a charged particle with more or less localized inhomogeneities of the magnetic field from slowly moving magnetic clouds mentioned by Fermi in his pioneering works [2] to strong shock waves in the supernovae remnants mentioned by Berezhko and Krymskii in review [3] can be considered as instantaneous jumps from one point of the momentum space to another one. The momenta $\Delta \mathbf{p}_i$ acquired by the particle in these collisions are random and, even for their isotropic distribution, the point

$$\mathbf{p} = \mathbf{p}_0 + \Delta \mathbf{p}_1 + \Delta \mathbf{p}_2 + \Delta \mathbf{p}_3 + \dots,$$

representing the particle in the momentum space moves away from the point (momentum) of the acceleration injection \mathbf{p}_0 similar to a Brownian particle; this behavior means the further acceleration (reacceleration) of the particle. However, only a certain fraction of the particles moving away from the center are accelerated. This fluctuation component of the mechanism of the acceleration of cosmic rays is analyzed in this work.

From the statistical point of view, the main consequence of the Fermi conclusion is that the exponential increase in the energy of the accelerated particle with time, $E = E_0 e^{at}$, and the exponential distribution of the age of the detected particles, $dP = \exp(-t/\tau)dt/\tau$,

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are sufficient for the formation of the power-law energy spectrum $N(E)$. That is all. There are no other sources of fluctuations taken into account in the Fermi model. This result is evident: What fluctuations can else exist if an increase in the energy by a factor of e requires 10^8 collisions according to the Fermi estimation? A more significant effect could be produced by more rare acts with more large acceleration in each of them. Among these processes are the aforementioned interactions with strong shock waves, when even a single interaction can increase the energy of the particle by a factor of $7 \div 13$ (see [4, p. 449]). To this end, it is appropriate to pass from the degenerate spectral function $\delta(E - E_0 e^{at})$ characterizing the determinate process of the Fermi acceleration to the continuous function $n(E, t)$ related to the momentum distribution $f(\mathbf{p}, t)$ as

$$n(E, t) = \int \delta(E - E(\mathbf{p})) f(\mathbf{p}, t) d\mathbf{p}.$$

Under the Fermi assumption that the parameter τ is independent of the energy, the desired spectrum can be represented in the form

$$N(E) \equiv N(E; \tau) = \tau^{-1} \left[\int_0^\infty n(E, t) e^{-t/\tau} dt \right] = \tau^{-1} \hat{n}(E, \tau^{-1}), \quad (3)$$

where

$$\hat{n}(E, \lambda) \equiv \int_0^\infty e^{-\lambda t} n(E, t) dt$$

is the Laplace transform of the spectral function in the time variable. The effect of the kinetic fluctuations is now taken into account at the stage of the construction of equations for the distributions $f(\mathbf{p}, t)$ or $n(E, t)$ by the inclusion of additional terms containing differential and integral operators. The theory of the acceleration of cosmic rays is under development and this work is focused on the role of fractional derivatives in this development.

2 Fractional kinetic equations

As was mentioned in [1], Eq. (1) is the diffusion limit of the coordinate (\mathbf{x} – CTRW) model, where the trajectory of a particle is a sequence of instantaneous jumps at random times by random distances; the particle is at rest between these times. The CTRW model with a finite velocity of free motion provides qualitatively new results (see, e.g. [5]). When analyzing the acceleration of cosmic rays in the first approximation, the stochastic dynamics of the particle in the velocity (momentum or energy) space is considered, and the boundedness of the coordinate space is taken into account by correcting the mean age τ or introducing additional coefficients characterizing the leakage of the particles from the acceleration region or from the Galaxy as a whole.

In the momentum space, the stationary position of the point that does not coincide with the coordinate origin means the motion of the particle with a constant momentum (velocity, energy). The exponent β characterizes now the tail part of the distribution of the random interval duration ΔT between the successive collisions of the moving particle $Q(t) = P(\Delta T > t)$, $t \rightarrow \infty$. For the ultrarelativistic particle ($v \approx c$), this interval is proportional to its mean free path between the successive collisions, so that the new β

values should correspond to the α values in the preceding $\mathbf{x} - CTRW$ model. In view of this circumstance, the consideration starts with the CTRW system of Eqs. (2)-(4) from [1], where the coordinates \mathbf{r} are changed by the momenta \mathbf{p} , the momentum change rate \mathbf{v} is accepted to be infinite, and it is assumed that the particle is injected by the source at the time $t = 0$ with the distribution function $f_0(\mathbf{p})$:

$$f(\mathbf{p}, t) = \int_0^t Q(t - t') F_{0 \leftarrow 1}(\mathbf{p}, t') dt', \quad (4a)$$

$$F_{0 \leftarrow 1}(\mathbf{p}, t) = \int w(\mathbf{p} \leftarrow \mathbf{p}') F_{1 \leftarrow 0}(\mathbf{p}', t) d\mathbf{p}' + f_0(\mathbf{p}) \delta(t), \quad (4b)$$

$$F_{1 \leftarrow 0}(\mathbf{p}, t) = \int_0^t q(t - t') F_{0 \leftarrow 1}(\mathbf{p}, t') dt'. \quad (4c)$$

The further transformation of the system is associated with the specification of the distributions $q(t)$ and $w(\mathbf{p} \leftarrow \mathbf{p}')$ distributions. The time is usually (one can say, always) taken in the exponential form $Q(t) = e^{-\mu t}$, $q(t) \equiv -dQ/dt = \mu e^{-\mu t}$ and therefore the master equation begins with the first-order time derivative $\partial f(\mathbf{p}, t)/\partial t = \dots$. This means that the process is assumed to be Markovian. However, the real distribution of the time intervals between collisions is unknown. For example, one can assume that it is of power law, $Q(t) \propto t^{-\alpha}$, rather than exponential. This is in agreement with the self-similar pattern of turbulent motions and with its power-type laws. The hypothesis of the fractal character of the interstellar medium [6] also provides power-law distributions. This concerns the behavior of magnetic field lines in interstellar space. They are usually represented as relatively smooth lines, which somewhere rest at the magnetic traps, intersect each other, sharply change their directions, and performing a "diffusion dance" in time. The leading centers of particles moving along spiral trajectories move along these lines. If these smooth sections become invisible and chaotic patterns of the structure become prevalent with the expansion of the field of view and the corresponding decrease in the scale, then this is an asymptotically homogeneous medium where the mean free paths can be simulated by a usual exponential function. If the expanding field of view includes increasing straight segments replacing the segments becoming small due to a decrease in the scale so that the structure remains (qualitatively) unchanged, this is a fractal structure. Under these conditions, the exponential distribution of mean free paths characteristic of a strongly mixing medium cannot be expected, but can not be completely rejected as well. The best compromise would be a family of distributions including both exponential and power-law distributions. Fortunately, such a family exists: it is a set of the functions

$$Q_\alpha(t) = E_\alpha(-\mu t^\alpha), \quad \alpha \in (0, 1],$$

where $E_\alpha(z) = \sum_{n=0}^{\infty} z^n / \Gamma(\alpha n + 1)$ are the Mittag-Leffler functions. The function $Q_\alpha(t)$ with $\alpha = 1$ is a usual exponential and with $\alpha < 1$ is a fractional exponential having a power-law asymptotic behavior $t^{-\alpha}$, $t \rightarrow \infty$. The corresponding density $q_\alpha(t)$ satisfies a fractional differential equation [7]; as a result, the entire system of Eqs. (4a)-(4c) has the fractional time-differential form

$${}_0D_t^\alpha f(\mathbf{p}, t) = \mu A f(\mathbf{p}, t) + f_0(\mathbf{p}) \delta_\alpha(t). \quad (5)$$

Here

$$Af(\mathbf{p}, t) = \int w(\mathbf{p} \leftarrow \mathbf{p}') f(\mathbf{p}', t) d\mathbf{p}' - f(\mathbf{p}, t) \quad (6)$$

is the acceleration integral with transitions density $w(\mathbf{p} \leftarrow \mathbf{p}')$. The time series of accelerating collisions forms a *fractional Poisson process of order α* [7], with $\alpha \rightarrow 1$ becoming an ordinary Poisson process which underlies classical kinetic equation (see (21.1) in [8]). Investigations performed in [9] indicate a qualitatively new property of this process: the average number of collisions increases more slowly ($\propto t^\alpha$) than in the usual case ($\propto t$) and relative fluctuations of the number of collisions in the limit $t \rightarrow \infty$ do not disappear, but tend to a limiting distribution depending on α (some sort of KNO scaling).

The transition from the first time derivative to the fractional derivative of the order $\alpha < 1$ does not require solving a fractional differential equation; it is more convenient to use the relation between the solutions of Eq. (5) with $\alpha < 1$ and $\alpha = 1$ [7],

$$f_\alpha(\mathbf{p}, t) = (t/\alpha) \int_0^\infty f_1(\mathbf{p}, \tau) g_+(t\tau^{-1/\alpha}; \alpha) \tau^{-1/\alpha-1} d\tau. \quad (7)$$

Here $g_+(t; \alpha)$ is the one-sided stable (in the Lévy sense) probability density determined by the Laplace transform

$$\int_0^\infty \exp(-\lambda t) g_+(t; \alpha) dt = \exp(-\lambda^\alpha),$$

and t, τ, λ are dimensionless variables. The Laplace transform of Eq. (7) in time with the use of the above formulas for the spectra provides the formula

$$N_\alpha(E; \tau) = N_1(E; \tau^\alpha). \quad (8)$$

It presents the effect of the fractal dimension of the fractional Poisson process of collisions $\alpha \in (0, 1]$ on the energy spectrum of cosmic rays: the spectrum $N_\alpha(E; \tau)$, formed by an ensemble of particles with the mean life time τ , which are accelerated according to the fractional Poisson law of the order $\alpha < 1$, coincides with the spectrum of particles that are accelerated by the usual Poisson process ($\alpha = 1$), but have the mean age τ (recall that the time is here dimensionless and the injection spectra $f_0(E)$ are the same in both problems). On the example of the Fermi spectrum, it is easy to see that the efficiency of acceleration decreases (the spectrum becomes steeper) with a decrease in the order of the process. The fractal character of the spatial distribution of accelerating regions naturally reduces the efficiency of acceleration.

3 Fractional Fokker-Planck equations

Similar to the classical case, the transition from the kinetic equation to the Fokker-Planck equation is associated with the transformation of the collision integral to a differential form by expanding the integrand into a series in the momentum increment to the second order terms. There are two variants of such an expansion, which provide slightly different equations (see, e.g., [8]). The first variant implies the smallness of the *absolute value of the change in the momentum* $|\Delta \mathbf{p}| = |\mathbf{p} - \mathbf{p}'|$, so that the momentum of the incident particle only slightly changes in the magnitude and direction in a single collision event (e.g., as in

the case of the collision of a heavy particle with a light one). The second variant implies the smallness of the *change in the absolute value of the momentum* $\Delta|\mathbf{p}| = |\mathbf{p}| - |\mathbf{p}'|$, whereas the change in its direction is not small and can have a wide distribution up to the isotropic one (in the case of the collision of a light particle with a heavy one). Under the assumption of the isotropic scattering, the fractional differential generalization of the Fokker–Planck equation is obtained in the form

$${}_0D_t^\alpha f(p, t) = \Delta_{\mathbf{p}}(K(p)f(p, t)) + f_0(p)\delta_\alpha(t), \quad (9)$$

where

$$K(p) = (\mu/2) \int (\Delta\mathbf{p})^2 w(\Delta\mathbf{p}; p) d\Delta\mathbf{p}$$

is the diffusivity in the momentum space. An energy analog of Eq. (9) ($\alpha = 1$) is well known in the physics of cosmic rays in the form (see Eq. (14.2) in [10])

$${}_0D_t^\alpha n(E, t) = \frac{\partial[a_1(E)n(E, t)]}{\partial E} + \frac{\partial^2[a_2(E)n(E, t)]}{\partial E^2} + n_0(E)\delta_\alpha(t). \quad (10)$$

At the same time, Eq.(9) with diffusion term

$$\Delta_{\mathbf{p}}(K(p)f(p, t)) = (\Delta_{\mathbf{p}}K(p))f(p, t) + 2(\nabla_{\mathbf{p}}K(p))\nabla_{\mathbf{p}}f(p, t) + K(p)\Delta_{\mathbf{p}}f(p, t)$$

significantly differs from another diffusion type equation (cm. (9.57), [10])

$${}_0D_t^\alpha f(p, t) = \nabla_{\mathbf{p}}(K(p)\nabla_{\mathbf{p}}f(p, t)) + f_0(p)\delta_\alpha(t). \quad (11)$$

The difference is due to the fact that Eq. (9) is derived in frame of the collision model, when the point presenting the particle instantaneously moves to another, possibly far, geometric point, violating the continuity of the trajectory in the momentum space, whereas the dynamic derivation of Eq. (11) implies that the trajectory in the momentum space is continuous and even differentiable. The classical versions (with $\alpha = 1$) of Eqs. (9)-(11) underlie the standard set of mathematical tools describing the fluctuation mechanisms of the acceleration of cosmic rays; their solutions are well known [4, 10, 11]. A common property of these models is the Gaussian character of the momentum distributions, which is due to the assumption that the second moment of the momentum acquired in an acceleration event, which enters into the diffusion coefficient, is finite and the acceleration rate is low. A change of the first time derivative in this equation to its fractional analog of the order $\alpha \in (0, 1)$ does not lead to an increase in the efficiency of acceleration; on the contrary, the acceleration rate in the subdiffusion regime decreases further. It is also noteworthy that the statement of some authors that diffusion at $\alpha > 1$ is accelerated is erroneous. This statement is based on linguistic intuition ("if $\alpha < 1$ means subdiffusion and $\alpha = 1$ corresponds normal diffusion, then $\alpha > 1$ should mean super diffusion") and is erroneous, because the solution $f(\mathbf{p}, t)$ at $\alpha > 1$ loses its probability meaning: it is not positively definite in this case. Thus, the transition to the fractional time derivative (with the unchanged remaining, diffusion part of the equation) as a method for enhancing the high energy part of the spectrum is physically unpromising. At the same time, the parameter α presents the existence of possible correlations in the spatial distribution of the acceleration regions and can be useful in this respect (recall that $\alpha = 1$ corresponds to the uniform Poisson distribution of such regions, which do not correlate with each other).

This parameter can be kept and the efficiency of the multiple acceleration mechanism can be increased only by modifying another operator of the equation, more precisely, by returning from the differential form of the acceleration operator, which describes a continuous slow collection of the energy, to the integral form describing acceleration as a sequence of events with large instantaneous (in terms of the considered "galactic" time scales) changes in the momenta. For this reason, it seems appropriate to change the momentum Laplacian to its fractional analog, because the fractional Laplacian contains a momentum integral operator with inverse power law kernel, which can ensure a high acceleration rate. Let us consider this variant in more detail.

The losses of the energy of a fast charged particle in a medium, which is described by equations similar to Eq. (10) (naturally, with $\alpha = 1$), cannot be larger than its initial energy and all of the moments of the energy loss are finite. In the problem of acceleration, there is no such definite limit of the energy increase; this fact is an additional reason to study the region with infinite dispersion, which attracts an increasing interest of researchers of anomalous diffusion processes. In this case, informative (in the asymptotic sense) results are obtained only when infinite dispersion is due to the power-law distributions:

$$\int_{|\Delta \mathbf{p}| > p} w(\Delta \mathbf{p}; \mathbf{p}') d\Delta \mathbf{p} \propto p^{-\gamma}, \quad p \rightarrow \infty. \quad (12)$$

If $\gamma > 2$, the second moment is finite and this corresponds to the classical diffusion region. If $\gamma < 2$, the second moment of the increment is infinite and this corresponds to the model of additive Levy flights. In this case, the equations for the momentum and energy distributions $f(p, t)$ and $n(E, t)$, respectively, following from the asymptotic CTRW analysis,

$${}_0D_t^\alpha f(p, t) = -K(-\Delta_{\mathbf{p}})^{\nu/2} f(p, t) + f_0(p) \delta_\alpha(t), \quad (13)$$

and

$${}_0D_t^\alpha n(E, t) = \begin{cases} \partial^\nu [a_\nu n(E, t)] / \partial E^\nu + n_0(E) \delta_\alpha(t), & 0 < \nu < 1; \\ \partial [a_1 n(E, t)] / \partial E + \partial^\nu [a_\nu n(E, t)] / \partial E^\nu + n_0(E) \delta_\alpha(t), & 1 < \nu < 2. \end{cases} \quad (14)$$

seem to be reasonable. Here

$$\nu = \begin{cases} \gamma, & \gamma \leq 2; \\ 2, & \gamma > 2, \end{cases}$$

$\partial^\nu / \partial E^\nu$ is the symbol closer to the ordinary notation of the fractional differential operator and K , a_1 and a_ν are the constant coefficients. The constancy of these coefficients is very significant for the derivation of the equations. Fractional differential equations are usually derived with the use of integral transformations whose applicability requires the constancy of the coefficients. It would be incorrect to derive, e.g., Eq. (13) in such a manner and, then, to place the variable diffusion coefficient $K(p)$ in front of the fractional Laplacian (this is obvious even on the example of Eqs. (9)–(11) with the integer Laplacian).

Equation (14) is clearly similar to the ν th term approximation of the expansion of a function that is zero at the reference point and it would seem that omitting the next (divergent!) term introduces an infinite error, but this is not the case. The reason is that the Taylor formula rather than the infinite Taylor series is used and the series expansion can be continued only until derivatives exist; after that, the remainder term, which is always finite, should be written. If the next derivative existed, its inclusion in the continued expansion would mean an approximation of the omitted term. If this derivative does not

exist, it cannot be used to approximate the omitted term and one should return to the initial point, where the derivative of the preceding order is used for approximation. The diffusion packet, which is described by Eq. (13) and propagates from the origin of the momentum coordinates, has the form of the three-dimensional isotropic fractional stable distribution $\psi_3^{(\nu, \alpha)}$, $\nu \in (0, 2]$, $\alpha \in (0, 1]$, smearing proportionally to $t^{\alpha/\nu}$:

$$f(p, t) = (Kt^\alpha)^{-3/\nu} \psi_3^{(\nu, \alpha)}((Kt^\alpha)^{-1/\nu} p).$$

Tails (or, more romantically, wings) of this distribution have a power-law form with the exponent ν . Physically, this means a peculiar leading effect: one of independent terms in the sum $\Delta \mathbf{p}_1 + \Delta \mathbf{p}_2 + \dots + \Delta \mathbf{p}_n$ is always outstanding and compared in magnitude with the reminder. This leading effect disappears at $\nu = 2$, when the distribution becomes Gaussian (sub-Gaussian). As a result, the spectrum at $\nu < 2$ has the form

$$N_1(E)dE \propto E^{-\nu-1}dE,$$

which is similar to the Fermi formula. The main difference is that the exponent ν in this case is independent of the age of the detected particles and is completely determined by the acceleration mechanism in an individual local event (collision). For this reason, the fractal character of the spatial distribution of accelerating regions also does not affect the slope of the resulting spectrum.

4 Integro-fractionally-differential model

A drawback of the model with the fractional momentum Laplacian is that the increments of the momentum in the acceleration event are independent of the momentum of the particle involved in the interaction, whereas in the Fermi model and in its later variants, the increments of the energy (and, therefore, the momentum) are on average proportional to the energy (momentum) of the particle before the interaction. In this case, the energy of the accelerated particle is expressed in terms of the product of independent random variables, rather than their sum. This model is called *multiplicative walk* in order to distinguish it from the *additive walk* model considered above. The increment of the momentum in the multiplicative model is proportional (in the statistical case) to the absolute value of the momentum p' of the particle coming into interaction,

$$\Delta \mathbf{p} = p' \mathbf{q}, \quad \int_{|\Delta \mathbf{p}| > p} w(\Delta \mathbf{p}; \mathbf{p}') d\Delta \mathbf{p} \propto (p/p')^{-\gamma}, \quad p \rightarrow \infty. \quad (15)$$

Under the assumption that the distribution of the proportionality vector \mathbf{q} is independent of \mathbf{p}' and isotropic,

$$W(\mathbf{q}; \mathbf{p}') d\mathbf{q} = (1/2)V(q)dq d\xi, \quad \xi = \cos(\mathbf{q}, \mathbf{p})$$

, kinetic equation (5) can be modified to the form

$${}_0D_t^\alpha f(p, t) = \mu \left\{ \int_{-1}^1 \frac{d\xi}{2} \int_0^\infty V(q) f\left(p / \sqrt{1 + 2\xi q + q^2}, t\right) / \left(\sqrt{1 + 2\xi q + q^2}\right)^3 dq - f(p, t) \right\} + f_0(p) \delta_\alpha(t), \quad (16)$$

representing a new model of a distributed reacceleration, more precisely, a new modification of the model proposed in [12, 13]. In order to make this model be closer to real processes of reacceleration, e.g., in the case of the intersection of shock fronts in the remnants of supernovae, we assume that [13]

$$V(q) = \gamma q^{-\gamma-1}, \quad \gamma > 1.$$

The resulting model can be called *multiplicative Levy flights*.

Let us consider the equation for the spectral function in two extreme cases. In the first case, $\gamma > 2$, the second moment of the momentum increment proportional to E^2 exists, and this case corresponds to the classical diffusion with variable coefficients:

$${}_0D_t^\alpha n(E, t) = \frac{\partial[a_1 E n(E, t)]}{\partial E} + \frac{\partial^2[a_2 E^2 n(E, t)]}{\partial E^2} + n_0(E) \delta_\alpha(t).$$

In the second case, we suppose $\gamma \ll 2$, so that only the term q^2 may be retained in the radicand in Eq. (16):

$${}_0D_t^\alpha n(E, t) = \mu \left\{ \int_1^\infty \gamma q^{-\gamma-1} n(E/q, t) dq/q - n(E, t) \right\} + n_0(E) \delta_\alpha(t). \quad (17)$$

I do not claim that this approximation is very good, but namely this acceleration operator was used in [13] for some calculations. Solving Eq. (17) with the method of the Mellin–Laplace transforms and using Eqs. (3) and (8), the following expression is obtained for the case of the mono-energetic source ($n(E) = \delta(E - E_0)$):

$$N_\alpha(E; \tau) = \frac{\mu \tau^\alpha \gamma}{(1 + \mu \tau^\alpha)^2} \left(\frac{E}{E_0} \right)^{-1 - \gamma/(1 + \mu \tau^\alpha)} \frac{1}{E_0}. \quad (18)$$

Although Eq. (18) was derived under the assumptions that strongly simplify the real situation and has a qualitative sense, it compactly presents the effect of *all three sources of fluctuation acceleration*: fluctuations of the age of the particle (parameter τ), fluctuations of the number of the acceleration events (α и μ), and fluctuations of the energy acquired in a single event (γ). Representing the scaling parameter μ in the form $\mu = \tau_A^\alpha$, where τ_A is the characteristic time interval between the events of the acceleration of particles in the remnants of various supernovae (recall that τ is the mean lifetime with respect to nuclear collisions), the absolute value of the exponent of the integral spectrum can be written in the more clear form $\gamma' = 1 + \gamma/[1 + (\tau/\tau_A)^\alpha]$. At $\alpha = 1$, and $\mu \tau \gg 1$, we arrive at the Fermi formula (2) with $a = \mu/\gamma$.

5 Conclusions

Taking into account that all three above-considered mechanisms contribute to the acceleration (more precisely, reacceleration) of cosmic rays, this process can be analyzed using the integro-fractional-differential equation

$${}_0D_t^\alpha f(p, t) = \mu_0 \nabla_{\mathbf{p}} (K_0(p) \nabla_{\mathbf{p}} f(p, t)) + \mu_1 \Delta_{\mathbf{p}} (K_1(p) f(p, t)) - \mu_2 K_2(-\Delta_{\mathbf{p}})^{\nu/2} f(p, t) +$$

$$\mu_3 \left\{ \int_{-1}^1 \frac{d\xi}{2} \int_0^\infty V(q) f \left(p / \sqrt{1 + 2\xi q + q^2}, t \right) / \left(\sqrt{1 + 2\xi q + q^2} \right)^3 dq - f(p, t) \right\} + f_0(p) \delta_\alpha(t). \quad (19)$$

The terms with the coefficients μ_0 , μ_1 , μ_2 and μ_3 successively present the contributions from the processes of continuous acceleration (such as fluctuations of the momentum of the particle in turbulent plasma [11, 14]), collision accelerations with a finite second moment (the model of walking magnetized clouds with a limited distribution of their velocities), additive Levy-type collision accelerations given by Eq. (12) (the same with the power-law distribution of the velocities of the clouds), and multiplicative Levy-type collision accelerations given by Eq. (15). Recall again that the last operator describing, in particular, acceleration at shock waves in the remnants of supernovae does not reduced to the fractional differential form, but holds in the integral form. Equation (19) should be supplemented by terms presenting energy losses, exist of particles from the Galaxy, and nuclear interactions (fragmentation).

However, only dominant terms can be retained in Eq. (19) applied to particular problems, as is usually done.

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